On Products of Random Matrices and certain Hecke Algebras associated with Groups of 2×2 Matrices

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Abstract

The determination of the density functions for products of random elements from specified classes of matrices is a basic problem in random matrix theory and is also of interest in theoretical physics. For connected simple Lie groups of 2×2 matrices and conjugacy and spherical classes a complete solution is given here. The problem/solution can be re-stated in terms of the structure of certain Hecke algebras attached to groups of 2×2 matrices.

AMS Subject Classification 2000: 20E45, 22E46, 43A90, 53C35. 20C08.

1 Introduction

Let (G, K) be a symmetric pair where G is a semi-simple Lie group with finite center. For G compact let $\mathcal{H}_c = \mathcal{H}_c(G)$ be the algebra under convolution generated by the invariant measures concentrated on conjugacy classes in G, and $\mathcal{H}_s = \mathcal{H}_s(G, K)$ be the convolution algebra generated by the invariant measures on spherical classes $\mathcal{O}_a = KaK \subset G$. It is elementary that $\mathcal{H}_c$ and $\mathcal{H}_s$ are commutative algebras with unit. These algebras arise naturally in random matrix theory, and theoretical physics. In fact, the product structures for generators of these algebras are given by the density functions of products of random matrices chosen according to invariant measures on conjugacy or spherical classes; and in string theory generic D-branes are localized along the product of (twisted) conjugacy classes of the Lie group. We will not discuss specific physical interpretations in this paper (except for a remark in §5), and the reader is referred to [Q], and references thereof, for a detailed discussion of the issues of interest in string theory.

This work may be regarded as the initial attempt at understanding the structure of the Hecke algebras $\mathcal{H}_c$ and $\mathcal{H}_s$ and considers only the special case of 2×2 matrices. The support of the density function for products of two generators of $\mathcal{H}_c(SU(n))$ is determined in [AW] and is exhibited by a set of linear inequalities on the Lie algebra of a maximal

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torus. The result in [AW] is essentially a reformulation of a theorem about singular connections on a holomorphic vector bundle on a Riemann surface with marked points [B], or equivalently a theorem of Mehta and Seshadri in algebraic geometry. Neither theorem is applicable for the computation of the density function for the product of two generators of the Hecke algebra.

A complete structure theorem for $\mathcal{H}_c(SU(2))$ is given in Theorem 2.1 below. For spherical classes attached to groups of 2×2 matrices we consider the symmetric pairs $(SU(2), S(U(1) \times U(1))$, $(SL(2,\mathbb{R}), SO(2))$ and $(SL(2,\mathbb{C}), SU(2))$. Theorem 2.2 gives density functions for products of spherical classes. In the final section we present some numerical results in relation to products of conjugacy classes in $SU(2)$. Each conjugacy class, with the induced metric, is a copy of S^2 equipped with a metric of constant curvature. It is noticed (perhaps surprisingly) that if S^2 is dicretized according to the prescription of Thomson's problem (minimizing Coulomb potential or intuitively "the best equally spaced distribution") then convergence to the predicted measure is much slower than if the points were chosen randomly.

In [JM] the question of whether a product of conjugacy classes $\mathcal{C}_{\alpha_1}, \ldots, \mathcal{C}_{\alpha_n}$ contains the identity element and/or is the entire group $SU(2)$ is studied. By successive applications of Theorem 2.1 of this paper one can recover the results in [JM] and in fact give a more precise version of it. This subject has not been elaborated on here. The argument in [JM] depends on the results in [AW], but the proofs presented here are self-contained and of a more elementary nature.

The author wishes to thank A. Katanforoush for the numerical results, R. Szabo, M. Sheikh-Jabbari, S. Shahshahani and especially Mehrdad Shahshahani for many stimulating discussions.

2 Statement of main results

A conjugacy class $C_\theta \subset SU(2)$ is uniquely determined by the eigenvalues $e^{\pm i\theta}$ of a matrix in C_θ. Each conjugacy class is a homogeneous space, and for C_θ this measure μ_θ is uniquely normalized to be $4\pi \sin^2\theta$ in accordance with Weyl's integration formula and the normalization vol.$(SU(2)) = 4\pi^2$.

Theorem 2.1 *Let μ_α and μ_β be the invariant measure on the conjugacy classes C_α and C_β respectively (regarded as singular distribution on G). Then $\mu_\alpha \star \mu_\beta$ is an absolutely continuous conjugation invariant measure on G relative to the Haar measure and its density is given by*

$$\mu_\alpha \star \mu_\beta = \begin{cases} 4\pi^2 \sin\alpha \sin\beta \sin\theta, & \text{for } \ \alpha - \beta \le \theta \le \alpha + \beta, \\ -4\pi^2 \sin\alpha \sin\beta \sin\theta, & \text{for } \ -\alpha - \beta \le \theta \le -\alpha + \beta, \\ 0, & \text{otherwise,} \end{cases} \tag{2.1}$$

where $0 < \beta \le \alpha < \pi$.

Let $K = S(U(1) \times U(1)) \subset SU(2)$, then (G, K) is a symmetric pair of compact type [H1]. Orbits of the action of $K \times K$ on G via

$$a \longrightarrow kak', \quad (k, k') \in K \times K, \quad a \in G$$

are called spherical classes and are denoted by $\mathcal{O}_a$. $\mathcal{O}_a$ is a homogeneous space and it is a simple calculation that $|a_{11}|$ is constant on $\mathcal{O}_a$ and uniquely determines it. Here $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SU(2)$ is any matrix in $\mathcal{O}_a$. Each spherical class $\mathcal{O}_a$ carries an invariant measure λ_a. The total mass of the measure λ_a of a spherical class $\mathcal{O}_a$ is $2|a_{11}|$, and we set $r(a) = |a_{11}|$.

In the non-compact cases $(SL(2,\mathbb{R}), SO(2))$ and $(SL(2,\mathbb{C}), SU(2))$ the real Cartan subgroup A is

$$A = \{a_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \mid t \in \mathbb{R}\}$$

$A_+ = \{a_t \in A \mid t > 0\}$ parametrizes spherical classes. The Haar measure for the Cartan (polar) decomposition $G \simeq KAK$ is

$$\int_G f(g)dg = c \int_K \int_K \int_0^\infty f(k_1 a k_2)\delta(t) dt dk_1 dk_2 \tag{2.2}$$

where c is a suitable constant (see [H2]), $\delta(t) = \sinh^2 t$, and $\epsilon = 1$ or 2 according as $G = SL(2,\mathbb{R})$ or $SL(2,\mathbb{C})$. The volume (area) of the spherical class $\mathcal{O}_{a_t}$ is

$$\text{vol.}(\mathcal{O}_{a_t}) = [4\pi^2 c \sinh t]^\epsilon. \tag{2.3}$$

Theorem 2.2 *The product formula for spherical classes in the three cases of symmetric pairs attached to groups of 2×2 matrices are:*
(A)- *Let λ_a and λ_b be two (singular) spherical measures concentrated on the spherical classes $\mathcal{O}_a$ and $\mathcal{O}_b$ respectively. Then $\lambda_a \star \lambda_b$ is absolutely continuous relative to the Haar measure on $SU(2)$ and given by*

$$\begin{cases} \frac{16\pi^2 |a_{11} b_{11}| u}{\sqrt{c_1^2 - (u^2 - c_0)^2}}, & \text{for } \sqrt{c_0 - c_1} \le u \le \sqrt{c_0 + c_1}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.4}$$

where c_0 and c_1 are symmetric functions of a and b and given by

$$\begin{aligned} c_0 &= r^2(a) r^2(b) + (1 - r^2(a))(1 - r^2(b)), \\ c_1 &= 2r(a)r(b)\sqrt{(1 - r^2(a))(1 - r^2(b))}. \end{aligned}$$

(B) - *Let* $\lambda_{a_{t_1}}$ *and* $\lambda_{a_{t_2}}$ *be the (singular) invariant measures concentrated on the spherical classes* $\mathcal{O}_{a_{t_1}}$ *and* $\mathcal{O}_{a_{t_2}}$ *in* $SL(2,\mathbb{R})$*. Then* $\lambda_{a_{t_1}} \star \lambda_{a_{t_2}}$ *is spherical and absolutely continuous relative to the Haar measure, and for a continuous spherical function* f *on* $SL(2,\mathbb{R})$ *we have*

$$\lambda_a \star \lambda_b(f) = 4c^2\pi^2 \sinh t_1 \sinh t_2 \int_{I_{t_1,t_2}} f(r) \frac{\sinh r}{\sqrt{c_2^2 - (c_1 - \cosh r)^2}}\, dr\ ,$$

where $I_{t_1,t_2} = [t_2 - t_1, t_2 + t_1]$, $c_1 = \cosh t_1 \cosh t_2$, *and* $c_2 = \sinh t_1 \sinh t_2$*.*
(C) - *Let* $\lambda_{a_{t_1}}$ *and* $\lambda_{a_{t_2}}$ *be the (singular) invariant measures concentrated on the spherical classes* $\mathcal{O}_{a_{t_1}}$ *and* $\mathcal{O}_{a_{t_2}}$ *in* $SL(2,\mathbb{C})$*. Then* $\lambda_{a_{t_1}} \star \lambda_{a_{t_2}}$ *is spherical and absolutely continuous relative to the Haar measure, and for a continuous spherical function* f *on* $SL(2,\mathbb{C})$ *we have*

$$\lambda_{a_{t_1}} \star \lambda_{a_{t_2}}(f) = 32c^2\pi^6 \sinh t_1 \sinh t_2 \int_{I_{t_1,t_2}} f(r) \sinh r dr\ ,$$

where $I_{t_1,t_2} = [t_2 - t_1, t_2 + t_1]$*.*

It may be of interest to normalize the measures on the spherical classes to probability measures and determine the corresponding empirical measure of products. For such a normalization the density functions determined in Theorem 2.2 become

$$\mathrm{A}: \ \frac{1}{2\pi} \frac{u}{\sqrt{c_1^2 - (u^2 - c_0)^2}}, \quad \mathrm{B}: \ \frac{1}{\pi} \frac{\sinh r}{\sqrt{c_2^2 - (c_1 - \cosh r)^2}}, \quad \mathrm{C}: \ \frac{\sinh r}{2 \sinh t_1 \sinh t_2}. \tag{2.5}$$

Remark 2.1 Let $\tilde{\mathcal{H}}_c$ and $\tilde{\mathcal{H}}_s$ denote the completions of $\mathcal{H}_c$ and $\mathcal{H}_s$ in the weak topology. Then in all cases considered $\tilde{\mathcal{H}}_c$ and $\tilde{\mathcal{H}}_s$ contain the corresponding L^1 space as a dense ideal. Furthermore, by the above analysis, for every pair of generators μ_1 and μ_2, we have $\mu_1 \star \mu_2 \in L^1$.

3 Proof of Theorem 2.1

First we show that $\mu_\alpha \star \mu_\beta$ is an L^p function for $p \leq 2$. The Fourier expansion of the singular measure μ_α is given by

$$\sum_{\rho \in \hat{G}} d_\rho \mathrm{Tr}\big[\rho(\mu_\alpha)\rho(g)\big], \tag{3.1}$$

where $\hat{G}$ is the set of irreducible representations of G and the Fourier transform of a measure μ is defined as

$$\rho(\mu) = \int \rho(g^{-1}) d\mu.$$

This series does not converge in the ordinary sense of convergence of series of functions since the measure μ_α is singular, however, it converges in the weak sense. The convolution product $\mu_\alpha \star \mu_\beta$ can be calculated from

$$\sum_{\rho \in \hat{G}} d_\rho \mathrm{Tr}\big[\rho(\mu_\alpha)\rho(\mu_\beta)\rho(g)\big]. \tag{3.2}$$

Since the character $\chi_\rho(g)$ is independent of $g \in C_\alpha$, and is given by $\frac{\sin(n+1)\alpha}{\sin\alpha}$ for ρ is the symmetric n^{th} representation of $G = SU(2)$,

$$\rho(\mu_\alpha) = \frac{4\sin^2\alpha}{d_\rho}\mathrm{Tr}(\rho(g))I = \frac{4\chi_\rho(C_\alpha)\sin^2\alpha}{d_\rho}I.$$

Applying the Plancherel theorem to $\mu_\alpha \star \mu_\beta$, we obtain

$$\begin{aligned} ||\mu_\alpha \star \mu_\beta||^2 &= \sum \frac{16\sin^2\alpha\sin^2\beta\chi_\rho(C_\alpha)\chi_\rho(C_\beta)}{d_\rho^4}d_\rho^2 \\ &= \sum_{n\geq 1}\frac{16}{n^2}\sin^2(n+1)\alpha\sin^2(n+1)\beta. \end{aligned}$$

This series converges absolutely and therefore $\mu_\alpha \star \mu_\beta$ is a square integrable function. From the Cauchy-Schwartz inequality it follows that $\mu_\alpha \star \mu_\beta$ is absolutely integrable.

Therefore the measure $\mu_\alpha \star \mu_\beta$ is a conjugation invariant function and it can be interpreted as the (defective) density function for the space of solutions c to the equation

$$abc = e, \quad \text{where } \ a \in C_\alpha, \ b \in C_\beta.$$

Since this density is conjugation invariant we represent it as the function $\nu(\theta)$ on the maximal torus of diagonal matrices given by

$$\nu(\theta) = ((\mu_\alpha \star \mu_\beta) \star \mu_\theta)(e).$$

Let $\mathcal{R}$ denote the left regular representation of G, and $\Delta_\alpha = 4\pi\sin^2\alpha$ be the factor appearing in Weyl integration formula for conjugacy invariant functions. The Fourier transform of a function ψ on G at a representation ρ is defined as

$$\rho(\psi) = \int \rho(x^{-1})\psi(x)dx.$$

Let d_ρ denote the dimension of the representation ρ, χ_ρ its character and $\hat{G}$ the space of (complex) irreducible representations of G. Taking Fourier transform and decomposing

the regular representation $\mathcal{R}$ of G in the usual way we obtain

$$\begin{aligned}
\nu(\theta) &= ((\mu_\alpha \star \mu_\beta) \star \mu_\theta)(e) \\
&= \frac{1}{\mathrm{vol.}(G)} \mathrm{Tr}\mathcal{R}((\mu_\alpha \star \mu_\beta) \star \mu_\theta) \\
&= \frac{1}{\mathrm{vol.}(G)} \sum_{\rho \in \hat{G}} d_\rho \mathrm{Tr}\rho((\mu_\alpha \star \mu_\beta) \star \mu_\theta) \\
&= \frac{1}{\mathrm{vol.}(G)} \sum_{\rho \in \hat{G}} d_\rho \frac{\Delta_\alpha \Delta_\beta \Delta_\theta}{d_\rho^3} \chi_\rho(\alpha)\chi_\rho(\beta)\chi_\rho(\theta)\mathrm{Tr}(I) \\
&= \frac{\Delta_\alpha \Delta_\beta \Delta_\theta}{\mathrm{vol.}(G)} \sum_{\rho \in \hat{G}} \frac{\chi_\rho(\alpha)\chi_\rho(\beta)\chi_\rho(\theta)}{d_\rho}.
\end{aligned}$$

Therefore

$$\nu(\theta) = 16\pi \sin^2\alpha \sin^2\beta \sin^2\theta \sum_{\rho \in \hat{G}} \frac{\chi_\rho(\alpha)\chi_\rho(\beta)\chi_\rho(\theta)}{d_\rho} \tag{3.3}$$

Irreducible representations of $G = SU(2)$ are determined by dimension $k \geq 2$. The character of the k-dimensional representation ρ_k is:

$$\chi_k(g) = \sum_{j=0}^{k} e^{(j-2)i\theta} = \frac{\sin(k+1)\theta}{\sin\theta}$$

where $e^{\pm i\theta}$ is the eigenvalue of the conjugacy class of g. Substituting in (3.3) we obtain

$$\nu(\theta) = 16\pi \sin\alpha \sin\beta \sin\theta \sum_{k=0}^{\infty} \frac{\sin(k+1)\alpha \ \sin(k+1)\beta \ \sin(k+1)\theta}{k+1}. \tag{3.4}$$

Now let $f(\theta)$ denote the function defined by (2.1). Since f is an even function its Fourier expansion is of the form $f(\theta) = \frac{a_0}{2} + \sum a_n \cos nx$, and

$$a_n = \frac{2}{\pi}[\frac{1}{n+1}\sin(n+1)\alpha \sin(n+1)\beta - \frac{1}{n-1}\sin(n-1)\alpha \sin(n-1)\beta]$$

Substituting in the Fourier expansion we obtain

$$f(\theta) = 8\pi \sin\alpha \sin\beta \sum_{m=0} (\frac{\sin(m+1)\alpha \sin(m+1)\beta}{m+1} - \frac{\sin(m-1)\alpha \sin(m-1)\beta}{m-1}) \cos m\theta.$$

Using the elementary identity $\cos m\theta - \cos(m+2)\theta = 2\sin(m+1)\theta \sin\theta$ the series becomes telescopic and simplifies to

$$f(\theta) = 16\pi \sin\alpha \sin\beta \sin\theta \sum_{m=0} \frac{\sin(m+1)\alpha \sin(m+1)\beta \sin(m+1)\theta}{m+1},$$

which is identical with (3.4). ■

Remark 3.1 In the above proof we made use of the fact that the measure $\mu_\alpha \star \mu_\beta$ is a conjugation invariant function that can be interpreted as the (defective) density function for the space of solutions c to the equation

$$abc = e, \quad \text{where} \;\; a \in C_\alpha,\; b \in C_\beta$$

and this density is given by (3.3). For the case of finite groups this formula is well-known ([S], p.68).

Remark 3.2 It is possible to prove Theorem 2.1 without the use of harmonic analysis and by integral formulae similar to those used for the proof of Theorem 2.2 below. However, it appears that the above argument is possibly generalizable to products of conjugacy classes in compact connected semi-simple Lie groups, but the one based on integral formulae is not.

4 Proof of Theorem 2.2

Since the proofs of (B) and (C) are essentially the same computation we prove (A) and (C) only. Introduce coordinates on $SU(2)$ by:

$$(\rho,\varphi,\psi) \longrightarrow \begin{pmatrix} \rho e^{i\varphi} & \sqrt{1-\rho^2}e^{-i\psi} \\ -\sqrt{1-\rho^2}e^{i\psi} & \rho e^{-i\varphi} \end{pmatrix}, \quad (\rho,\varphi,\psi) \in [0,1]\times[0,2\pi]\times[0,2\pi] \tag{4.1}$$

The Haar measure on $SU(2)$ in the (ρ,φ,ψ)- coordinates is easily calculated by computing $g^{-1}dg$, a basis of left invariant 1-forms $\omega_1,\omega_2,\omega_3$ and then taking their wedge product to obtain:

$$\Omega = \omega_1 \wedge \omega_2 \wedge \omega_3 = 2\rho d\rho d\varphi d\psi.$$

With this normalization $\mathrm{vol}(SU(2)) = 4\pi^2$ as before.

Since both f and λ_a are K-bi-invariant, $\lambda_a \star f(x)$ is K-bi-invariant and therefore to compute $\mu_a \star f(x)$ we can assume that x is of the form

$$x = \begin{pmatrix} t & \overline{w} \\ -w & t \end{pmatrix} \tag{4.2}$$

where t is a real number and $w = se^{i\alpha}$ a complex number with $t^2+s^2=1$. In (ρ,φ,ψ)-coordinates on $SU(2)$ we have

$$\lambda_a \star \check{f}(x) = \int_{\mathcal{O}_a} f(yx^{-1})dy = 2r(a)\int_0^{2\pi}\int_0^{2\pi} f(y(\rho,\varphi,\psi)x^{-1})d\varphi d\psi$$

With x represented as in (4.2) we have

$$\begin{aligned}
\lambda_a \star \check{f}(x) &= 2r(a)\int_0^{2\pi}\int_0^{2\pi} f\left(\begin{pmatrix} \rho e^{i\varphi} & \sqrt{1-\rho^2}e^{-i\psi} \\ -\sqrt{1-\rho^2}e^{i\psi} & \rho e^{-i\varphi} \end{pmatrix}\begin{pmatrix} t & \overline{w} \\ -w & t \end{pmatrix}\right) d\varphi d\psi \\
&= 2r(a)\int_0^{2\pi}\int_0^{2\pi} f\left(\begin{pmatrix} r(a)te^{i\varphi} + b\sqrt{1-\rho^2}e^{-i\psi} & \star \\ \star & \star \end{pmatrix}\right) d\varphi d\psi
\end{aligned}$$

Since f is spherical, it depends only on the norm of the $(1,1)$ entry of the above matrix. The square of the norm of the $(1,1)$ entry is

$$t^2\ r^2(a) + s^2(1-r^2(a)) + 2tsr(a)\sqrt{1-r^2(a)}\cos(\varphi+\psi-\alpha).$$

Substituting $t = r(x)$ and $s = \sqrt{1-r^2(x)}$, the norm of the $(1,1)$ entry becomes

$$|r(a)te^{i\varphi} + b\sqrt{1-\rho^2}e^{-i\psi}| = \sqrt{c_0 + c_1\cos(\varphi+\psi-\alpha)}.$$

Therefore

$$\lambda_a \star \check{f}(x) = 2r(a)\int_0^{2\pi}\int_0^{2\pi} f(\sqrt{c_0 + c_1\cos(\varphi+\psi-\alpha)})\ d\varphi\ d\psi.$$

The change of variable

$$(u,v) = (\sqrt{c_0 + c_1\cos(\varphi+\psi-\alpha)}, \psi),$$

is a 2 to 1 covering. Its Jacobian is given by

$$\frac{\partial(\varphi,\psi)}{\partial(u,v)} = \frac{2u}{c_1\sin(\varphi+\psi-\alpha)} = \frac{2u}{\sqrt{c_1^2-(u^2-c_0)^2}}$$

Therefore

$$\begin{aligned}
\lambda_a \star \check{f}(x) &= 4\pi r(a)\int_0^{2\pi}\int_{\sqrt{c_0-c_1}}^{\sqrt{c_0+c_1}} f(u)\frac{u}{\sqrt{c_1^2-(u^2-c_0)^2}}\ dudv \\
&= 2r(a)\int_{\sqrt{c_0-c_1}}^{\sqrt{c_0+c_1}} f(u)\frac{u}{\sqrt{c_1^2-(u^2-c_0)^2}}\ du.
\end{aligned}$$

To compute $\lambda_a \star \lambda_b(f)$ we set $g(x) = \mu_b \star \check{f}(x)$. Then g is spherical and

$$\begin{aligned}
\lambda_a \star \lambda_b(f) &= \lambda_a \star (\lambda_b \star \check{f})(e) \\
&= (\lambda_a \star g)(e) \\
&= \int_{\mathcal{O}_a} g(x) d\lambda_a(x) \\
&= g(a)\mathrm{vol.}(\mathcal{O}_a),
\end{aligned}$$

Now

$$\text{vol.}(\mathcal{O}_a) = 2r(a) \int_0^{2\pi} \int_0^{2\pi} \, d\varphi \; d\psi = 8\pi^2 r(a).$$

Therefore

$$\mu_a \star \mu_b(f) = 8\pi^2 r(a) g(a).$$

Substituting from the calculation of $g(a) = \lambda_b \star \check{f}(a)$ above we obtain

$$\lambda_a \star \lambda_b(f) = g(a) \; \text{vol.}(\mathcal{O}_a) = 16\pi^2 r(a) r(b) \int_{\sqrt{c_0 - c_1}}^{\sqrt{c_0+c_1}} f(u) \; \frac{u}{\sqrt{c_1^2 - (u^2 - c_0)^2}} \; du$$

This completes the proof of part (A).
Proof of part (C) - Since both f and λ_a are K-bi-invariant, $\lambda_a \star f(x)$ is K-bi-invariant and therefore to compute $\mu_a \star f(x)$ we can assume that x is of the form $x = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}$. As before let $\{\theta_n\}$ be a sequence of spherical functions on G converging weakly to the (singular) invariant measure λ_a on the orbit $\mathcal{O}_a$. Applying the polar coordinate decomposition, for the convolution $\lambda_a \star \check{f}(x)$ we have

$$\begin{aligned}
\lambda_a \star \check{f}(x) &= \int_{\mathcal{O}_a} f(yx^{-1}) dy \\
&= \lim_{n\to\infty} \int_G \theta_n(g) f(gx^{-1}) dg \\
&= c \lim_{n\to\infty} \int_K \int_K \int_A \theta_n(k_1 a' k_2) f(k_1 a' k_2 x^{-1}) \delta(a') da' dk_1 dk_2 \\
&= c \lim_{n\to\infty} \int_K \int_A \theta_n(a') f(a' k x^{-1}) \delta(a') da' dk \\
&= c\delta(t_1) \int_K f(akx^{-1}) dk
\end{aligned}$$

Writing $M = akx^{-1} = k_1 a_1 k_2$, where $k_1, k_2 \in K = SU(2)$ and using the coordinates in (4.1)

$$a_1 = \begin{pmatrix} e^{\frac{r}{2}} & 0 \\ 0 & e^{\frac{-r}{2}} \end{pmatrix}, \quad k = \begin{pmatrix} \rho e^{i\varphi} & \sqrt{1-\rho^2} e^{-i\psi} \\ -\sqrt{1-\rho^2} e^{i\psi} & \rho e^{-i\varphi} \end{pmatrix},$$

we compute r in term of t_1, t_2 and k:

$$2\cosh r = \text{Tr}(a_1^2) = \text{Tr}(k_1 a_1^2 k_1^{-1}) = \text{Tr}(MM^*). \qquad (4.3)$$

On the other hand we have

$$M = akx^{-1} = \begin{pmatrix} e^{\frac{1}{2}(t_1+t)}\rho e^{i\varphi} & e^{\frac{1}{2}(t_1-t)}\sqrt{1-\rho^2}e^{-i\psi} \\ -e^{\frac{1}{2}(t-t_1)}\sqrt{1-\rho^2}e^{i\psi} & e^{-\frac{1}{2}(t_1+t)}\rho e^{-i\varphi} \end{pmatrix}$$

Therefore

$$\mathrm{Tr}(MM^*) = 2\rho^2\cosh(t+t_1) + 2(1-\rho^2)\cosh(t-t_1) \ ,$$

Comparing with (4.3) we obtain after a simple calculation

$$\cosh r = \cosh t_1 \cosh t - (2\rho^2 - 1)\sinh t_1 \sinh t$$

Now set

$$c_1 = \cosh t_1 \cosh t, \ c_2 = \sinh t_1 \sinh t$$

The function f is spherical so it only depends on the component r and therefore

$$\lambda_a \star \check{f}(x) = c\delta(t_1)\int_K f(akx^{-1})dk = c\delta(t_1)\int_0^{2\pi}\int_0^{2\pi}\int_0^1 f(r)(2\rho)d\rho d\varphi d\psi$$

Now make the change of coordinate $\cosh r = c_1 - c_2(2\rho^2 - 1)$ and note that, assuming that $t > t_1$, r ranges over $I_{t,t_1} = [t - t_1, t + t_1]$ as ρ ranges over $[0, 1]$. Substituting in the above integral for $\lambda_a \star \check{f}(x)$ we obtain

$$\lambda_a \star \check{f}(x) = \frac{2\pi^2 c}{c_2}\delta(t_1)\int_{I_{t,t_1}} f(r)\sinh r dr \ , \tag{4.4}$$

To compute $\lambda_a \star \lambda_b(f)$ we set $g(x) = \mu_b \star \check{f}(x)$. Then g is spherical and

$$\begin{aligned} \lambda_a \star \lambda_b(f) &= \lambda_a \star (\lambda_b \star \check{f})(e) \\ &= (\lambda_a \star g)(e) \\ &= \int_{\mathcal{O}_a} g(x)d\lambda_a(x) \\ &= g(a)\mathrm{vol.}(\mathcal{O}_a), \end{aligned}$$

Using (2.2) one easily obtains

$$\mathrm{vol.}(\mathcal{O}_{a_t}) = c(\mathrm{vol.}(SU(2)))^2\sinh^2 t = 16\pi^4\sinh^2 t. \tag{4.5}$$

Substituting from (4.5) we obtain

$$\lambda_a \star \lambda_b(f) = 16c\pi^4(\sinh t_1)^2 g(a).$$

Equation (4.4) implies

$$\lambda_a \star \lambda_b(f) = 32c^2\pi^6 \sinh t_1 \sinh t_2 \int_{I_{t_1,t_2}} f(r)\sinh r dr \ ,$$

which completes the proof of the theorem. ■

5 Discretization and Numerical Results

The discretization and quantization of conjugacy classes are essentially different issues, and the latter is based on the Kirillov orbit method (see [K]). It is of interest in theoretical physics and is also relevant to the subject matter of this paper. The dual of the Lie algebra of $SU(2)$ is identified with the set of 2×2 skew hermitian matrices of trace 0:

$$\xi(a,b,c) = \begin{pmatrix} ic & a+ib \\ -a+ib & -ic \end{pmatrix}$$

Under conjugation action of G the orbits are the spheres $\Sigma_r = \{\xi(a,b,c) \mid a^2+b^2+c^2 = r^2\}$. It is customary in physics to assign the symmetric n^{th} power representation ρ_n of $SU(2)$ to the sphere Σ_r of area $n \in \mathbb{Z}_+$. According to the Clebsch-Gordon formula, $\rho_n \otimes \rho_m$ decomposes as

$$\rho_n \otimes \rho_m \simeq \sum_{|n-m|}^{n+m} \rho_k, \quad \text{where} \;\; k \equiv m+n \mod 2.$$

On the other hand the Minkowski sum of spheres of radii r_1 and r_2 in $\mathbb{R}^3$ is precisely the spherical shell defined by $|r_2-r_1| \leq ||\xi|| \leq r_1+r_2$. The perfect resemblance between sums of spheres and the decomposition of the tensor product is carried over to the multiplication of conjugacy classes. In fact, we set (note slight change of notation)

$$\mathcal{C}_n = \exp(\Sigma_{\frac{n}{4\pi}}).$$

Let $\frac{n}{4\pi} \equiv \alpha \mod \pi$ and $\frac{m}{4\pi} \equiv \beta \mod \pi$. By Theorem 2.1

$$\mathcal{C}_n.\mathcal{C}_m = \exp(\Sigma_{\frac{n}{4\pi}} + \Sigma_{\frac{m}{4\pi}})$$

The representations corresponding to conjugacy classes contained in $\mathcal{C}_n.\mathcal{C}_m$ correspond to those integers $k \in [|m-n|, m+n]$ which $\equiv \; m+n \mod 2$. Thus in the range $[|m-n|, m+n]$ about half the representation occurring in $\mathcal{C}_n.\mathcal{C}_m$ appear in the Clebsch-Gordon formula. This is what is meant by the quantization of the product of two conjugacy classes.

By the discretization of products of conjugacy classes $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$ one means the choice of N points on each and the determination of the corresponding empirical measure of products of these points. If these N points are chosen randomly according to the invariant measures on the conjugacy classes then the empirical measure of the products converges weakly to the density function (2.1) by Theorem 2.1 and is numerically demonstrated in Figures 1 for a typical choice with $Np = 2172$ points.

Each conjugacy class $\mathcal{C}_\alpha$ is naturally equivalent to a copy of S^2. It is therefore reasonable to investigate the weak convergence of the empirical measure if the points on S^2 are

chosen according to the requirements of Thomson's Problem of distributing points on the sphere (see [KS]). This means that the points should be distributed so that the Coulomb potential

$$\sum_{i<j}^{N} \frac{1}{|z_i - z_j|^\alpha}$$

is minimized. A variation of this problem for $\alpha = 1$ was originally posed by J. J. Thomson in connection with his investigations of the structure of the atom in 1904. It remains unsolved except for a few small values of N, and it has also attracted attention for applications to complexity theory [Sm]. The lattice point method makes use of the natural embedding of the icosahedron in S^2 and distributes $N = 10(mn + m^2 + n^2) + 2$ points on the sphere in such a way that the distribution exhibits a high degree of symmetry and the points appear to be "evenly" spaced. It was conjectured in [A] that this distribution will provide the solution to Thomson's Problem for $\alpha = 1$. The polar coordinates method was devised in [KS] to achieve the minimum required by Thomson's Problem and numerical tests disproved the conjecture in [A] by showing that the (local) minimum achieved by the polar coordinates method (where there was symmetry breakdown) was in fact smaller. It is therefore natural to test the convergence of the empirical measure of products if the discretization is done according the polar coordinates or the lattice point methods. In Figures 1 and 2 the convergence of the empirical measure to the density predicted by Theorem 2.1 is exhibited for $Np = 2172$ points. The angles ϕ and ψ in the captions refer to the conjugacy classes $\mathcal{C}_\phi$ and $\mathcal{C}_\psi$ respectively. In Figures 2 the corresponding empirical measures are calculated for the lattice point method and it is noticed that even if the measure converges to the density given by (2.1), the convergence is significantly slower. Similar conclusion is applicable to the polar coordinates method as shown in Figures 3.

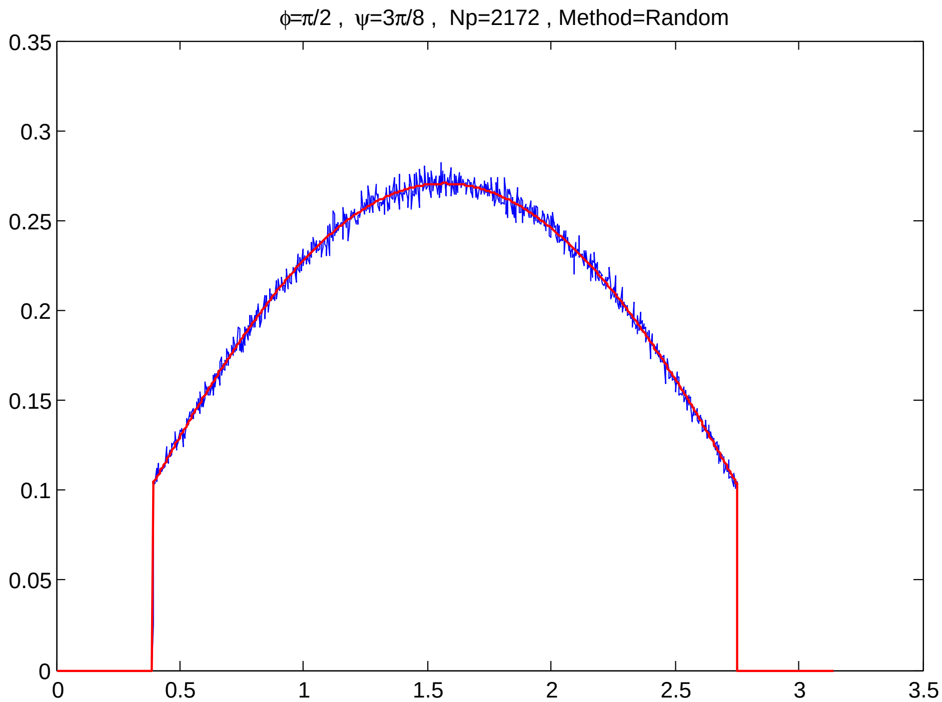

Figure 1.

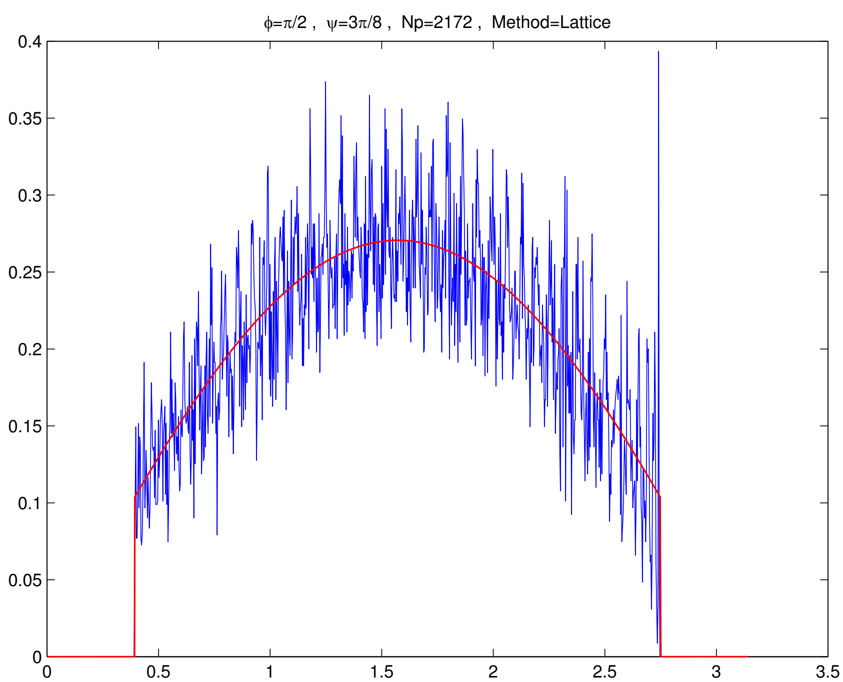

Figure 2.

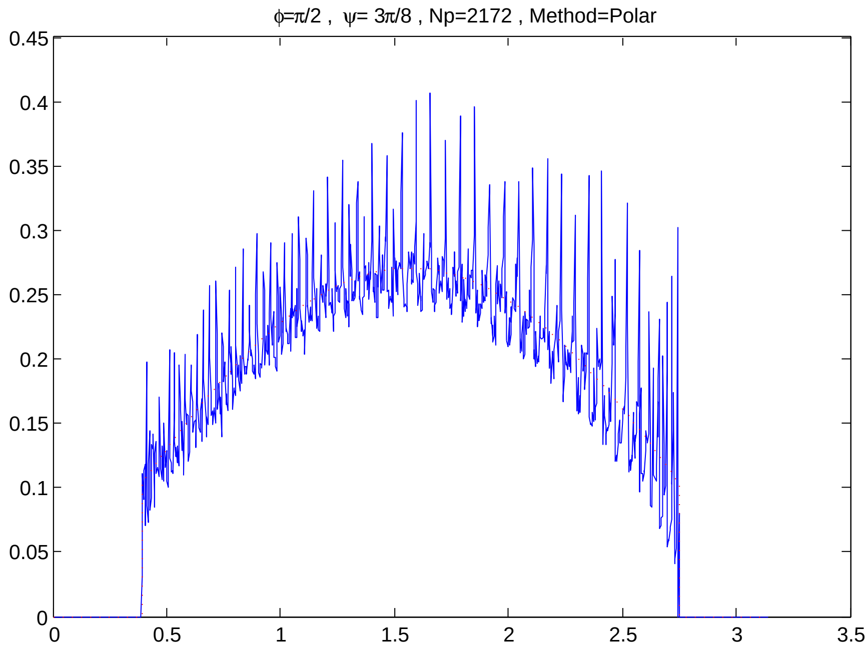
ϕ=π/2 , ψ= 3π/8 , Np=2172 , Method=Polar
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.5
1
1.5
2
2.5
3
3.5

Figure 3.

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